Int. J. Solids Structures, 1967, Vol. 3, pp. 795 to 798. Pergamon Press Ltd. Printed in Great Britain

NATURE OF SINGULARITY FOR A POINT LOAD AT THE EDGE OF A PLATE IN BENDING

J. DUNDURS

Northwestern University, Evanston, Illinois

and

M. STIPPES

University of Illinois, Urbana, Illinois

Abstract—Starting with the Lur'e solution for the bending of a circular plate with a free edge, it is shown that the singularity of a point load, applied at the edge of a plate, depends on the curvature of the boundary.

A FEW examples are known in theory of elasticity for the somewhat unusual situation of the singularity for a concentrated load depending on the curvature of the boundary [1–3]. The purpose of the present note is to show that this also is the case with a *thin elastic plate loaded in bending*, when the deflection satisfies the equation $\nabla^4 w = q/D$. The new example is particularly simple because we can start with a known solution for the load applied at an interior point, so that there only remains to explore the nature of the singularity as the load is moved to the edge of the plate.

The solution for a point load on a *circular* plate with a free edge was given by Lur'e [4] in 1940. Using the coordinates and some other geometric parameters shown in Fig. 1, a slightly modified form of the deflection function is

$$
\begin{aligned}
w = \frac{Pa^2}{8\pi D}\Bigg\{ & \frac{r_1^2}{a^2}\left(\log\frac{r_1}{a}+\frac{1-\nu}{3+\nu}\log\frac{\beta r_2}{a}\right) \\
& +\frac{8(1+\nu)}{(3+\nu)(1-\nu)}\left[\frac{\beta r_2}{a}\left(\theta_2\sin\theta_2-\log\frac{r_2}{a}\cos\theta_2\right)+S\right] \\
& -\frac{1}{2(1+\nu)}\left[3+\nu-\frac{(1-\nu)^2\beta^2}{3+\nu}\right]\frac{r^2}{a^2}+\frac{2\beta}{3+\nu}\cdot\frac{r^3}{a^3}\cos\theta\Bigg\},
\end{aligned}
\tag{1}
$$

where

$$
S = \mathscr{R}\left\{\int_0^{\beta z/a} t^{-1}\log(1-t)\,\mathrm{d}t\right\} = -\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{\beta r}{a}\right)^n\cos n\theta, \tag{2}
$$

and $\mathscr{R}$ stands for the real part. Furthermore, D denotes the flexural rigidity of the plate, and ν is Poisson's ratio. The angle θ_2 is restricted to the interval $0 < \theta_2 < 2\pi$.

The boundary conditions, satisfied by w on $r = a$, are

$$
M_r = 0, \qquad V_r = \frac{P}{a}\left[-\frac{1+\beta\cos\theta}{2\pi}+H(\beta-1)\delta(\theta)\right]. \tag{3}
$$

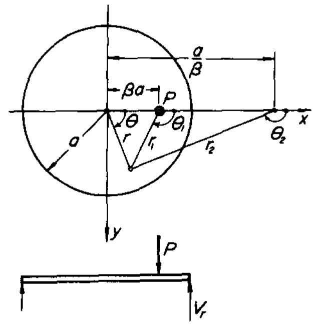

FIG. 1

Here M_r and V_r are the radial bending moment and supplemented (Kirchhoff) shearing force, respectively, H is the Heaviside step-function, and δ denotes the Dirac delta function. It is seen from (3) that, if (1) were used to generate by superposition a solution for loads forming an equilibrium system and no loads were applied at the edge of the plate, the resulting deflection function will yield $M_r = V_r = 0$ at $r = a$. If, however, some point loads were applied directly at the edge, the supplemented shearing force would contain the appropriate delta functions instead of vanishing everywhere.

In the limit $\beta \to 1$, we have $r_2 \to r_1$, $\theta_2 \to \theta_1$, and there is no difficulty in extracting the singular part of the elementary functions in (1). The quantity S, however, requires some special considerations. Introducing

$$z_1 = z - a, \tag{4}$$

and denoting the integral in (2) for $\beta = 1$ by I, it follows that

$$I = \int_0^{1+(z_1/a)} t^{-1} \log(1-t)\, dt. \tag{5}$$

Next, changing the variable of integration to $u = 1-t$,

$$I = -\int_1^{-z_1/a} (1-u)^{-1} \log u\, du, \tag{6}$$

and as an additive constant in I is of no consequence, this is equivalent to

$$I = -\int_0^{-z_1/a} (1-u)^{-1} \log u\, du. \tag{7}$$

Since we are interested only in the conditions near the point of load application, or $z_1 = 0$, we can put in (7)

$$(1-u)^{-1} = 1 + u + u^2 + u^3 + \cdots \tag{8}$$

Thus,

$$I = \frac{z_1}{a}\left[\log\left(-\frac{z_1}{a}\right) - 1\right] - \frac{1}{2}\left(\frac{z_1}{a}\right)^2\left[\log\left(-\frac{z_1}{a}\right) - \frac{1}{2}\right]$$
$$+ \frac{1}{3}\left(\frac{z_1}{a}\right)^3\left[\log\left(-\frac{z_1}{a}\right) - \frac{1}{3}\right] - \frac{1}{4}\left(\frac{z_1}{a}\right)^4\left[\log\left(-\frac{z_1}{a}\right) - \frac{1}{4}\right] + \cdots \tag{9}$$

The terms in the deflection function that give unbounded or discontinuous stress resultants at the point of load application ($x = a$, $y = 0$) will be called the singular part of deflection, or simply singularity, and denoted by w_s. As the shearing forces involve third derivatives of deflection, it is seen that three terms must be kept in I. Substituting these into (2) and discarding all regular functions, the singular part of S is

$$S_s = \frac{r_1}{a}(\log r_1 \cos\theta_1 - \theta_1 \sin\theta_1) - \frac{r_1^2}{2a^2}(\log r_1 \cos 2\theta_1 - \theta_1 \sin 2\theta_1) + \frac{r_1^3}{3a^3}(\log r_1 \cos 3\theta_1 - \theta_1 \sin 3\theta_1). \tag{10}$$

The final result is obtained by combining S_s with the other singular terms from (1). It is convenient, however, to write this in terms of the angle $\theta_1'(-\pi/2 \leq \theta_1' \leq \pi/2)$ shown in Fig. 2, so that the singularity is automatically symmetric about the diameter passing through the load point. Thus,

$$w_s = \frac{P}{2\pi(3+\nu)D}\left\{r_1^2 \log r_1 - \frac{1+\nu}{1-\nu}\left[r_1^2(\log r_1 \cos 2\theta_1' - \theta_1' \sin 2\theta_1') + \frac{2r_1^3}{3a}(\log r_1 \cos 3\theta_1' - \theta_1' \sin 3\theta_1')\right]\right\} \tag{11}$$

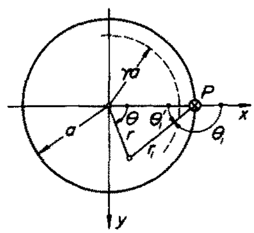

FIG. 2

The dependence of the singularity on curvature of the boundary, or $1/a$, is brought out by the last term in (11). It may be noted that for a straight edge ($1/a = 0$) w_s agrees with the singularity given by Nádai [5].

It is interesting to trace through the contributions to the *singular* parts of boundary data, viz. M_r and V_r at $r = a$, by the individual terms in w_s. The proper approach that allows one to extract the δ-function content of the boundary data is to first compute these quantities on the circle $r = \gamma a$, and then to take the limit $\gamma \to 1$. The details of such a calculation are quite tedious and, therefore, we quote only the final result where *all continuous terms have been discarded.*

For

$$w = r_1^2 \log r_1,$$
$$M_r/D = -(1+\nu)\log(1-\cos\theta),$$
$$aV_r/D = 4\pi\delta(\theta).$$

For

$$w = r_1^2(\log r_1 \cos 2\theta_1' - \theta_1' \sin 2\theta_1'),$$
$$M_r/D = -(1-\nu)\cos 2\theta \log(1-\cos\theta),$$
$$aV_r/D = 2(1-\nu)[\cos 2\theta \log(1-\cos\theta) - \pi\delta(\theta)].$$

For

$$w = (r_1^3/a)(\log r_1 \cos 3\theta_1' - \theta_1' \sin 3\theta_1'),$$
$$M_r/D = 3(1-\nu)(\cos 3\theta - \cos 2\theta)\log(1-\cos\theta),$$
$$aV_r/D = -3(1-\nu)(3\cos 3\theta - 2\cos 2\theta)\log(1-\cos\theta).$$

It follows from these expressions that w_s leads to the following boundary data:

$$(M_r)_{r=a} = f(\theta),$$
$$(V_r)_{r=a} = \frac{P}{a}\delta(\theta) + g(\theta),$$

where $f(\theta)$ and $g(\theta)$ are finite and continuous at the load point.

There is a significant difference regarding the physical implications, however, between the known examples in theory of elasticity, when the singularity depends on curvature, and the present case. In the former, the terms in the singularity that depend on curvature correspond to stresses that decay more slowly away from the load point than those for the boundary with zero curvature. In contrast, the bending and twisting moments derived from the third term in w_s from (11) are continuous at the load point. Furthermore, the third term in w_s is harmonic, and it does not contribute to the resultants of *transverse* shearing stresses, or the shearing forces Q_x and Q_y. Consequently, there are no logarithmic terms in these quantities, and they still decay as $1/r_1$ away from the load. The curvature-dependent term in w_s gives logarithmic contributions only in the supplemented shearing force V_r which involves a derivative of the twisting moment.

REFERENCES

[1] E. STERNBERG and F. ROSENTHAL, The elastic sphere under concentrated loads. *J. appl. Mech.* **19**, 413 (1952).
[2] E. STERNBERG and R. A. EUBANKS, On the singularity at a concentrated load applied to a curved surface. *Proc. 2nd U.S. natl. Congr. appl. Mech.* 237 (1954).
[3] J. DUNDURS, K. FUKUI and T. FUKUI, Concentrated force near a smooth circular inclusion. *J. appl. Mech.* **33**, 871 (1966).
[4] A. I. LUR'E, Some problems in bending of a circular plate (in Russian). *Prikl. Mat. Mekh.* **4**, 93 (1940).
[5] A. NÁDAI, *Elastische Platten*. Springer (1925).

(*Received* 28 *December* 1966)

Résumé—Partant de la solution de Lur'e pour le pliage d'une plaque circulaire à bord libre, il est démontré que la singularité d'une charge sur un point, appliquée au bord de la plaque, dépend de la courbure de la délimitation.

Zusammenfassung—Ausgehend von der Lur'eschen Lösung der Biegung einer kreisförmigen Platte mit freiem Rand, wird gezeigt, dass die Singularität einer Punktbelastung am Rande einer Platte von der Krümmung der Grenze abhängt.

Абстракт—Выходя из решения Лурье для изгиба цилиндрической пластинки со свободным краем показано, что сингулярность точечной нагрузки приложенная на краю пластинки, зависит от кривизны контура.